Part 2: Question 1 (20 points)

Consider an economy with $I = 2$ consumers and $L = 2$ goods, where $X_i = \mathbb{R}^2_+$ for $i = 1, 2$. The preferences for each consumer can be represented by the following utility functions:

$$u_1(x_{11}, x_{21}) = x_{11} - \frac{1}{8}x_{21}^{-8}$$
$$u_2(x_{12}, x_{22}) = -\frac{1}{8}x_{12}^{-8} + x_{22}$$

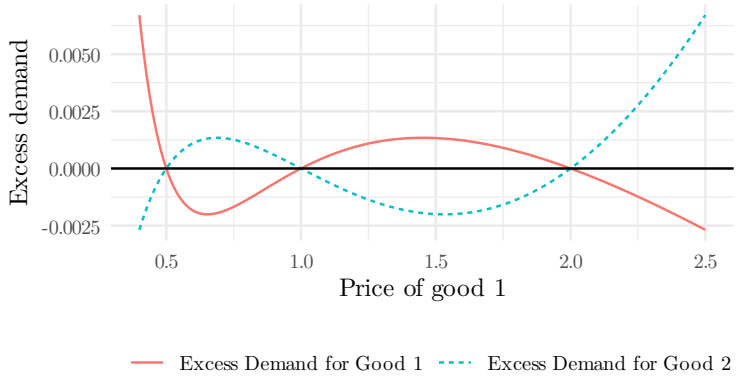
The initial endowment vectors are $\boldsymbol{\omega}_1 = \left(2, 2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right)$ and $\boldsymbol{\omega}_2 = \left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}, 2\right)$. There is a single firm whose only production possibilities are free disposal: $Y_1 = -\mathbb{R}^2_+$. Each consumer owns a 50% stake in the firm. In this question, we normalize the price of good 2 to $p_2 = 1$.

Note: $2^{\frac{1}{9}} \approx 1.08006$ and $2^{\frac{8}{9}} \approx 1.85175$, so $2^{\frac{8}{9}} - 2^{\frac{1}{9}} \approx 0.77169 < \frac{7}{9} \approx 0.77778$.

(i) **[6 points]** Derive the aggregate demand function $\boldsymbol{z}(p_1)$ for this economy. Show that it is equal to:

$$z_1(p_1) = \left(\frac{1 - p_1}{p_1}\right)\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - p_1^{-\frac{8}{9}} + p_1^{-\frac{1}{9}}$$
$$z_2(p_1) = p_1^{\frac{1}{9}} + (p_1 - 1)\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - p_1^{\frac{8}{9}}$$

A graph of $z_1(p_1)$ and $z_2(p_1)$ for different values of p_1 is shown below:



(ii) **[4 points]** Show mathematically (i.e. not just referring to the graph) that $\boldsymbol{p}' = \left(\frac{1}{2}, 1\right)$, $\boldsymbol{p}''(1, 1)$ and $\boldsymbol{p}''' = (2, 1)$ are all equilibrium price vectors.

(iii) **[3 points]** Derive $\frac{\partial z_1(p_1)}{\partial p_1}$ and evaluate it at $p_1 = 1$. Show that it is positive.

(iv) **[2 points]** Does $\boldsymbol{z}(p_1)$ satisfy the gross-substitute property? Show whether it does or does not.

(v) **[2 points]** Is this economy regular? If so, what is its index? You may base your answer on the graph above.

(vi) **[3 points]** Suppose when determining the equilibrium prices of this economy that prices adjust over time according to: $\frac{dp_1(t)}{dt} = -z_1\left(p_1\left(t\right)\right)$. Which equilibria are locally stable? Is there system stability? You may base your answer on the graph above.

Part 2: Question 2 (20 points)

Consider an economy with 2 consumers, 1 firm, and L goods. Each consumer $i = 1, 2$ has the consumption set $X_i = \mathbb{R}^L_+$ and has a preference relation $\succeq_i$ satisfying completeness, transitivity, convexity, continuity, and local nonsatiation. The single firm's production set Y_1 is convex and exhibits free disposal. The aggregate initial endowment vector satisfies $\bar{\boldsymbol{\omega}} \gg \mathbf{0}$.

In this economy, there exists an allocation $(\boldsymbol{x}_1^\star, \boldsymbol{x}_2^\star, \boldsymbol{y}_1^\star)$ with $\boldsymbol{x}_i \gg \mathbf{0}$ for all i that is Pareto optimal.

(i) **[5 points]** Define $V_i = \{\boldsymbol{x}_i \in X_i : \boldsymbol{x}_i \succ_i \boldsymbol{x}_i^\star\}$. Show that V_i is a convex set. Do this by taking two arbitrary elements in V_i and showing that their convex combination is also contained in V_i.

(ii) **[5 points]** Define the set:

$$V = \left\{\boldsymbol{x}_1 + \boldsymbol{x}_2 \in \mathbb{R}^L : \boldsymbol{x}_1 \in V_1 \text{ and } \boldsymbol{x}_2 \in V_2\right\}$$

This is the set of aggregate consumption bundles that can be split across the two consumers leaving them both strictly better off compared to the Pareto optimal allocation $(\boldsymbol{x}_1^\star, \boldsymbol{x}_2^\star, \boldsymbol{y}_1^\star)$.
Show that the set V is convex. Do this by taking two arbitrary elements in V and showing that their convex combination is also contained in V.

(iii) **[5 points]** Define the set:

$$B = \left\{\boldsymbol{y}_1 + \bar{\boldsymbol{\omega}} \in \mathbb{R}^L : \boldsymbol{y}_1 \in Y_1\right\}$$

Show that B is convex. Do this by taking two arbitrary elements of B and showing their convex combination is also contained in B.

(iv) **[5 points]** Show that the sets V and B are disjoint. *Hint:* Use proof by contradiction.